Faculty of Engineering
EM 509 – Stochastic Processes
Answer Sheet
Transform Methods in Stochastic Theory

1. If the characteristic function of X is $\psi_X(t)$,then the characteristic function of $a+bX$,where a and b are constants.

$$\psi_{a+bX}(t)=E\left(e^{it(a+bX)}\right)=E\left(e^{iat}.e^{itbX}\right)=E\left(e^{iat}\right)E\left(e^{itbX}\right)=e^{iat}E\left(e^{itbX}\right)=e^{iat}\psi_X(bt)$$

2. The characteristic function of a random variable $Z\in N(0,1)$ can be written as $\psi_Z(t)=e^{-t^2/2}$ and the random variable $X\in N(\mu,\sigma^2)$ is given by $X=\mu+\sigma Z$.
Using the property given in question 1,
$\psi_X(t)=e^{i\mu t}\psi_Z(\sigma t)=e^{i\mu t}e^{-\sigma^2t^2/2}=e^{i\mu t-\sigma^2t^2/2}$

3. If X has Bernoulli distribution with probability of success p, The characteristic function can be written as,
$\psi_X(t)=E\left(e^{itX}\right)=(1-p)e^{it0}+pe^{it}=(1-p)+pe^{it}$

4. Suppose $X_1,X_2,...,X_n$ are independent random variables and each $X_k\in N(\mu_k,\sigma_k^2);\ k=1,2,...,n$.The characteristic function for any constants $a_1,a_2,...,a_n$ can be written as $\psi_{a_kX_k}(t)=E\left(e^{ita_kX_K}\right)=e^{i\mu_ka_kt-\sigma_k^2a_k^2t^2/2}$

Let us consider, the characteristic function of $S_n=\sum_{k=1}^{n}a_kX_k$;

$$\begin{aligned}\psi_{S_n}(t)&=\prod_{k=1}^{n}\psi_{a_kX_k}(t)\\&=\prod_{k=1}^{n}e^{i\mu_ka_kt-\sigma_k^2a_k^2t^2/2}\\&=e^{\sum_{k=1}^{n}\left(i\mu_ka_kt-\sigma_k^2a_k^2t^2/2\right)}\end{aligned}$$

Hence , $S_n=\sum_{k=1}^{n}a_kX_k\in N\left(\sum_{k=1}^{n}a_k\mu_k,\sum_{k=1}^{n}a_k^2\sigma_k^2\right)$

If $X_1, X_2, \ldots, X_n$ are iid and $N(\mu, \sigma^2)$, using above result :

$$\overline{X} = \sum_{k=1}^{n} \frac{1}{n} X_k \in N\left(\sum_{k=1}^{n} \frac{1}{n} \mu, \sum_{k=1}^{n} \frac{1}{n^2} \sigma^2 \right)$$

$$= \frac{1}{n} \sum_{k=1}^{n} X_k \in N\left(\mu, \frac{\sigma^2}{n} \right)$$