1. If the characteristic function of $X$ is $\varphi_X(t)$, then the characteristic function of $a+bX$, where $a$ and $b$ are constants,

$$\varphi_{a+bX}(t) = E(e^{i(a+bX)t}) = E(e^{iat}e^{ibt}) = E(e^{iat})E(e^{ibt}) = e^{iat}E(e^{ibt}) = e^{iat} \varphi_X(bt)$$

2. The characteristic function of a random variable $Z \sim N(0,1)$ can be written as $\varphi_Z(t) = e^{-t^2/2}$ and the random variable $X \sim N(\mu, \sigma^2)$ is given by $X = \mu + \sigma Z$.

Using the property given in question 1,

$$\varphi_X(t) = e^{i\mu t} \varphi_Z(\sigma t) = e^{i\mu t} e^{-\sigma^2t^2/2} = e^{i\mu t - \sigma^2t^2/2}$$

3. If $X$ has Bernoulli distribution with probability of success $p$, the characteristic function can be written as,

$$\varphi_X(t) = E(e^{itX}) = (1-p)e^{it0} + pe^{it} = (1-p) + pe^{it}$$

4. Suppose $X_1, X_2, \ldots, X_n$ are independent random variables and each $X_k \sim N(\mu_k, \sigma_k^2); k = 1, 2, \ldots, n$. The characteristic function for any constants $a_1, a_2, \ldots, a_n$ can be written as $\varphi_{a_1X_k}(t) = E(e^{ita_kX_k}) = e^{i\mu_k a_k t - \sigma_k^2a_k^2t^2/2}$

Let us consider, the characteristic function of $S_n = \sum_{k=1}^{n} a_k X_k$ ;

$$\varphi_{S_n}(t) = \prod_{k=1}^{n} \varphi_{a_kX_k}(t)$$

$$= \prod_{k=1}^{n} e^{i\mu_k a_k t - \sigma_k^2a_k^2t^2/2}$$

$$= e^{i\mu S_n - \sigma^2S_n^2/2}$$

Hence, $S_n = \sum_{k=1}^{n} a_k X_k \sim N\left(\sum_{k=1}^{n} a_k \mu_k, \sum_{k=1}^{n} a_k^2 \sigma_k^2\right)$
If $X_1, X_2, \ldots, X_n$ are iid and $N(\mu, \sigma^2)$, using above result:

$$
\bar{X} = \frac{1}{n} \sum_{k=1}^{n} X_k \in N \left( \frac{1}{n} \sum_{k=1}^{n} \mu, \frac{1}{n} \sum_{k=1}^{n} \frac{1}{n} \sigma^2 \right) \\
= \frac{1}{n} \sum_{k=1}^{n} X_k \in N \left( \mu, \frac{\sigma^2}{n} \right)
$$