1) First mode: Random angle

Take an arbitrary point P on the boundary of the disc. The set of all lines through that points are parameterized by an angle $\phi$. The line has to lie within a sector of $60^\circ$ within a range of $180^\circ$.

The probability of this chord intersects the concentric circle of radius $1 = \frac{60^\circ}{180^\circ} = \frac{1}{3}$

Second mode: Random translation

Take all lines perpendicular to a fixed diameter. If the point of intersection lies on the middle half of the diameter.

The probability of this chord intersects the concentric circle of radius $1 = \frac{1}{2}$

Third mode: Random area

If the midpoints of the chords lie in a disc of radius $1/2$, because the disc has a radius which is half the radius of the unit disc.

probability of this chord intersects the concentric circle of radius $1 = \frac{\pi (1/2)^2}{\pi 2^2} = \frac{1}{4}$
2) a) Direct measure
\[ F = P(\Omega) \quad x_0 \in \Omega \ 	ext{fixed} \]
\[ \delta_{x_0} = \begin{cases} 
1; x_0 \in A \\
0; x_0 \notin A 
\end{cases} \]

I. \[ \delta_{x_0} \geq 0 \text{ for } A \in F \]
II. \[ \delta_{x_0}(\phi) = 0 \]
III. \[ A \text{ and } A^c \text{ are disjoint events} \]
\[ \delta_{x_0}(A \cup A^c) = \delta_{x_0}(\Omega) = 1 = \delta_{x_0}(A) + \delta_{x_0}(A^c) \]

Therefore, \( \delta_{x_0} \) is a measure.

b) Counting measure
\[ F = P(\Omega), \mu(A) = \text{Cardinality of } A = |A| \]

I. \[ \mu(A) \geq 0 \text{ for all } A \in F \]
II. \[ \mu(\phi) = 0 \]
III. Consider the following 3 cases and let \( \{A_1, A_2, \ldots, A_j, \ldots\} \) be a countable set of disjoint elements for \( j \in \mathbb{N} \)

Case I
Assume that all \( A_j \)'s are finite and only finitely many \( A_j \)'s are non-empty
\[ \sum_{j \in \mathbb{N}} \mu(A_j) = \sum_{j \in \mathbb{N}} |A_j| = \left| \bigcup_{j \in \mathbb{N}} A_j \right| = \mu \left( \bigcup_{j \in \mathbb{N}} A_j \right) \]

Case II
Assume that the cardinality of at least one \( A_j \) is infinite such that \( \mu(A_k) = \infty \)
\[ \mu \left( \bigcup_{j \in \mathbb{N}} A_k \right) = \infty = \mu(A_k) + \sum_{j \in \mathbb{N}, j \neq k} \mu(A_j) = \sum_{j \in \mathbb{N}} \mu(A_j) \]

Case III
Assume that infinitely many \( A_j \)'s are non-empty. The cardinality of \( \bigcup_{j \in \mathbb{N}} A_j \) is infinite due to \( A_j \)'s are disjoint.
\[ \mu \left( \bigcup_{j \in \mathbb{N}} A_k \right) = \infty = \sum_{j \in \mathbb{N}} \mu(A_j) \]

Therefore, \( \mu(A) \) is a measure.
Suppose \( i \in \{1, 2, \ldots, n\} \) denote a given channel and \( \varepsilon_i = 1 \) if \( i \) is active and \( \varepsilon_i = 0 \) if \( i \) is failing.

The sample space \( \Omega \) can be written as,
\[
\Omega = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) ; \varepsilon_i \in \{0,1\} \forall i\}
\]

Let \( A_k \) denote the event in which the communication rate is \( \rho k \),
\[
A_k = \{\omega \in \Omega, \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n = k\}
\]

Hence, \( F : \) consists of all union of \( A_k \),

The probability measure \( P \) can be formulated as a Binomial distribution such that,
\[
P(A_k) = \binom{n}{k} \rho^k(1-\rho)^{n-k} ; \ 0 < \rho < 1
\]

4) a) The sample space \( \Omega \) can be written as,
\[
\Omega = \{\omega : \omega = (d_0d_1\ldots d_n ; d_i \in \{0,1\})\}
\]

\( F = P(\Omega) = \) All the power set = \( 2^\alpha \)

\[
P(A) = \frac{\text{Cardinality of } A}{\text{Cardinality of } \Omega} = \frac{|A|}{256} \text{ for any event } A \in F
\]

b) Consider the following events:

\( E_1 = \) No two neighboring digits are the same
\[
= \{01010101,1010101\}
\]

\[
P(E_1) = \frac{2}{256} = \frac{1}{128}
\]

\( E_2 = \) Some cyclic shift of the register contains is equal to 01100110
\[
= \{01100110 \ 00110011 \ 100110011\}
\]

\[
P(E_2) = \frac{4}{256} = \frac{1}{64}
\]

\( E_3 = \) The register contains exactly 4 zeros
\[
= \{\omega \in \Omega ; d_0 + d_1 + \ldots + d_7 = 4\}
\]

\[
P(E_3) = \binom{8}{4} = \frac{70}{256} = \frac{35}{128}
\]
\(E_4 = \) There is a run of at least 6 consecutive ones
\[= \{01111110,01111111,11111000,11111110,11111100,11111111,10111111,11111111,00111111,01111111\}\n\[P(E_4) = \frac{8}{256} = \frac{1}{32}\]

c) \(P(E_1 / E_3) = \frac{P(E_1 \cap E_3)}{P(E_3)} = \frac{P(E_1)}{P(E_3)} = \frac{1}{35}\)
\[P(E_2 / E_3) = \frac{P(E_2 \cap E_3)}{P(E_3)} = \frac{P(E_2)}{P(E_3)} = \frac{2}{35}\]

5) The sample space \(\Omega\) can be written as,
\[\Omega = \{w : w = (w_1, w_2, \ldots), w_i \in \{H, T\} \forall i\}\]
Let \(E_i\) be the event \(i^{th}\) flip comes up heads
\[E_i = \{w \in \Omega; w_i = H\}, P(E_i) = p\]

a) Heads are got for the first time exactly on the \(n^{th}\) flip = \(\{X(w) = n\}\)
\[\{X = n\} = \{w = (T, T, T, \ldots, \underbrace{H, \ldots}_{n-1\text{times}})\}\]
\[= E_1^c \cap E_2^c \cap \ldots \cap E_{n-1}^c \cap E_n\]
\[P(\{X = n\}) = P(E_1^c \cap E_2^c \cap \ldots \cap E_n)\]
\[= P(E_1^c)P(E_2^c)\ldots P(E_{n-1}^c)P(E_n)\]
\[= (1 - p)^{n-1}p\]

b) Heads are got for the first time at time 6 or later.
\[\{X \geq 6\} = \{w = (T, T, T, T, \ldots)\text{ can be } H \text{ or } T\}\]
\[= E_1^c \cap E_2^c \cap \ldots \cap E_5^c\]
\[P(\{X \geq 6\}) = P(E_1^c \cap E_2^c \cap \ldots \cap E_5^c)\]
\[= P(E_1^c)P(E_2^c)\ldots P(E_5^c)\]
\[= (1 - p)^5\]

c) Heads are got for the first time at the least by time 5 = \(\{X \leq 5\}\)
\[P(\{X \leq 5\}) = P(\{X \geq 6\}^c)\]
\[= 1 - P(\{X \geq 6\})\]
\[= 1 - (1 - p)^5\]