EM 509: Stochastic Processes

Class Notes

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Properties of a Stochastic Process

- In the previous lecture notes we saw various types of characterizations of a random process. Each of these characterizations contains statistical information concerning the process.

- There are several properties of random processes that make it easier to specify such characterizations.

- Of particular importance are properties describing conditions where the nature of the process randomness does not change with time.

- Relationships between forms of statistical stationarity exits and will be found useful for describing further statistical properties of random variables.

Properties of a Stochastic Process

- **Stationarity of a Stochastic Process**

1. **Strict sense stationary(SSS):**

   A stochastic process $X(t)$ is called strict sense stationary(strictly stationary) or strong sense stationary(strongly stationary) if the joint distribution of any collection of samples depends only on their relative time. That is, for any n, ϵ and for any $t_1, t_2, \cdots, t_n$ the set of random variables $X(t_1), X(t_2), \cdots, X(t_n)$ and $X(t_1 + \epsilon), X(t_2 + \epsilon), \cdots, X(t_n + \epsilon)$ have the same distribution,

$$f(X_1, \cdots, X_n; t_1, \cdots, t_n) = f(X_1, \cdots, X_n; t_1 + \epsilon, \cdots, t_n + \epsilon)$$
$$p(X_1, \cdots, X_n; t_1, \cdots, t_n) = p(X_1, \cdots, X_n; t_1 + \epsilon, \cdots, t_n + \epsilon)\,.$$

For n=1it is called first order stationary, for n=2 it is called second order stationary and for n=L it is called Lth order stationary. If it is stationary for all orders then it is called sss.

Properties of a Stochastic Process

There are several consequences of strictly stationarity.

- Since $f_X(x,t) = f_X(x,\tau)$ **or** $p_X(x,t) = p_X(x,\tau), \forall t, \tau$; the mean and the variance of the process must be independent of time.

  For mean $\mu_X(t) = \int x f_X(x,t)dx = \int x f_X(x,\tau) = \mu_X(\tau)$. This is true for all t, τ . $\therefore \mu_X(t) = \mu_X$.

  Similarly $\sigma_X^2(t) = \sigma_X^2 \ \forall t$.

- The second-order joint density/mass functions depend only on the difference of the time indices and not on individual times t_1, t_2 .

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; 0, t_2 - t_1) \text{ and } p_X(x_1, x_2, t_1, t_2) = p_X(x_1, x_2; 0, t_2 - t_1) .$$

- Such processes has second-order time-shift-invariant statistics. For example the autocorrelation has the property $R_{XX}(t_1, t_2) = R_{XX}(t_1 + \epsilon, t_2 + \epsilon) = R_{XX}(0, t_2 - t_1)$. Therefore second-order moment functions such as autocorrelations and autocovariances depend only on the differences in t. That is,

$$R_{XX}(t_1, t_2) = R_{XX}(0, t_2 - t_1) \equiv R_{XX}(t_2 - t_1)$$
$$C_{XX}(t_1, t_2) = C_{XX}(0, t_2 - t_1) \equiv C_{XX}(t_2 - t_1) .$$

The difference $t_2 - t_1$ is called **lag**.

Properties of a Stochastic Process

2. Wide Sense Stationary(WSS):

A process $X(t)$ is said to be wide sense stationary or weakly stationary if the mean of the process does not depend on t(that is a constant) and the autocorrelation depends only on the time difference.

$\mu_X(t) = \mu_x$ **stationarity in mean**

$R_{XX}(t_1, t_2) = R_{XX}(t_2 - t_1)$ **stationarity in autocorrelation** .

3. Periodic:

A stochastic process $X(t)$ is said to be periodic if the joint probability density/mass function is invariant when the time of any of the variables is shifted by integer multiples of some period T. That is, for any k, for any integers m_i and sampling times $t_1, \cdots, t_k$,

$$f_X(x_1, \cdots, x_k; t_1, \cdots, t_k) = f_X(x_1, \cdots, x_k; t_1 - m_1T, \cdots, t_k - m_kT)$$
$$p_X(x_1, \cdots, x_k; t_1, \cdots, t_k) = p_X(x_1, \cdots, x_k; t_1 - m_1T, \cdots, t_k - m_kT) .$$

Properties of a Stochastic Process

4. **Cyclostationary:**

A stochastic process $X(t)$ is said to be cyclostationary if the joint probability density/mass function is invariant when the time origin is shifted by integer multiples of some period T. That is, for any k, for any integer m and sampling times $t_1, \cdots, t_k$,

$$f_X(x_1, \cdots, x_k; t_1, \cdots, t_k) = f_X(x_1, \cdots, x_k; t_1 - mT, \cdots, t_k - mT)$$
$$p_X(x_1, \cdots, x_k; t_1, \cdots, t_k) = p_X(x_1, \cdots, x_k; t_1 - mT, \cdots, t_k - mT)\,.$$

- The difference between periodic and cyclostationary is that, in periodic processes, each time index can be shifted by a different multiple of the period, while in cyclostationary processes, all time indices must receive the same shift.

- Note that periodic implies cyclostationary, but not the converse.

Properties of a Stochastic Process

5. Wide Sense Periodic:

A stochastic process $X(t)$ is said to be wide sense periodic if the mean and autocorrelation of the process are invariant when the time of any of the variables is shifted by integer multiples of some period T. That is, for any k, for any integers m, m_1, m_2 and sampling times t_1, t_2

$$\mu_X(t) = \mu_X(t - mT),\ R_{XX}(t_1, t_2) = R_{XX}(t_1 - m_1T, t_2 - m_2T)\ .$$

6. Wide Sense Cyclostationary:

A stochastic process $X(t)$ is said to be wide sense cyclostationary if the mean and autocorrelation of the process are invariant when the time origin is shifted by integer multiples of some period T. That is, for any k, for any integer m and sampling times t_1, t_2

$$\mu_X(t) = \mu_X(t - mT),\ R_{XX}(t_1, t_2) = R_{XX}(t_1 - mT, t_2 - mT)\ .$$

Properties of a Stochastic Process

Examples:

1.For $X_t = \cos(2\pi ft + \Theta)$(example 1 in the previous note) we saw $\mu_X = 0$ and

$$R_{XX}(t_1, t_2) = \cos(2\pi f(t_1 - t_2))/2.$$

Thus autocorrelation satisfies $R_{XX}(t_1, t_2) = R_{XX}(t_2 - t_1)$ .Therefore process is WSS.

2.For the process $X_n = Z_1 + Z_2 + \cdots + Z_n, \;\; n = 1,2,\cdots$ with $Z_i's$ uncorrelated with zero mean and variance $\sigma^2, \forall i$ (example 2 in the previous note), we saw that the variance is $n\sigma^2$ which pends on n. Therefore the process is not sss.

3. Consider a real-valued harmonic process $X_n = A\sin(n\omega_0 + \phi)$ where ω_0, ϕ are fixed constants, but the amplitude A is a random variable that is uniformly distributed over the interval [b, c] with c>b. Determine the stationarity of the random process.

The mean of the process,

$$\mu_X(n) = E(X_n) = E[A\sin(n\omega_0 + \phi)] = E[A]\sin(n\omega_0 + \phi) = \frac{b+c}{2}\sin(n\omega_0 + \phi)$$

which depends on n. Therefore, a harmonic process with random amplitude is not WSS.